Mechanism Design

A little bit of algebra

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September 30, 2024

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- Content extracted from Auction Theory [1]

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- $f(\mathbf{x}) = f_1(x_1) \times f_2(x_2) \times \cdots \times f_N(x_N)$, where $\mathbf{x} = (x_1, x_2, \ldots, x_N)$

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- $\mathbf{x}_{-i} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$

Direct Mechanism

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- **Big assumption:** We assume that $\mathbf{x}$ is the truthfully values of buyers... $\mathbf{x}$ is the equilibrium for buyers. *why?*

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- Where q_i is the probability that i will get the object when he reports his value to be z_i and all other buyers report their values truthfully. Finally, m_i is the expected payoff of i when his report is z_i and all others buyers tell the truth.

cont.

- The expected payoff of buyer i when his true value is x_i and he reports z_i, assuming that all other buyers tell the truth is:

$$q_i(z_i)x_i - m_i(z_i)$$

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- The direct mechanism $(\mathbf{Q}, \mathbf{M})$ is said to be incentive compatible if:

$$u_i(x_i) \equiv q_i(x_i)x_i - m_i(x_i) \geq q_i(z_i)x_i - m_i(z_i)$$

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- Incentive compatibility implies that:

$$u_i(x_i) = \max_{z_i \in \mathcal{X}_i} \{q_i(z_i)x_i - m_i(z_i)\}$$

cont.

- For all x_i and z_i:

$$\begin{aligned} q_i(x_i)z_i - m_i(x_i) &= q_i(x_i)x_i - m_i(x_i) + q_i(x_i)(z_i - x_i) \\ &= u_i(x_i) + q_i(x_i)(z_i - x_i) \\ u_i(z_i) &\geq u_i(x_i) + q_i(x_i)(z_i - x_i) \end{aligned} \tag{IC}$$

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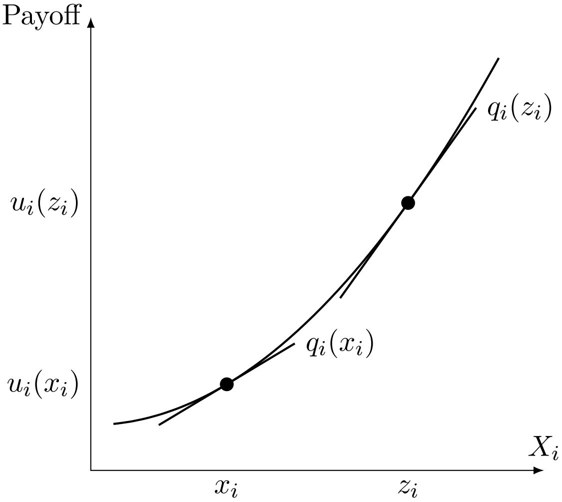
- Using the theorem of the envelope:

$$\begin{aligned} \frac{\partial}{\partial x_i} u_i(x_i) &= \frac{\partial}{\partial x_i} \max_{z_i \in \mathcal{X}_i} \{q_i(z_i)x_i - m_i(z_i)\} \\ &= q_i(x_i) \end{aligned}$$

cont.



cont.

- Because $u'(x_i) = q_i(x_i)$:

$$u_i(x_i) = u_i(0) + \int_0^{x_i} q_i(s_i) ds_i$$

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- Using $u(z_i) \geq u(x_i) + q_i(x_i)(z_i - x_i)$:

$$\int_{x_i}^{z_i} q_i(s_i) ds_i \geq q_i(x_i)(z_i - x_i)$$

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- Incentive compatibility holds q_i non decreasing.

Revenue equivalence and Individually Rational

- If $(\mathbf{Q}, \mathbf{M})$ is incentive compatible, then for all i and x_i the expected payment is:

$$m_i(x_i) = m_i(0) + q_i(x_i)x_i - \int_0^{x_i} q_i(s_i)ds_i$$

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- A mechanism is individually rational if $u_i(x_i) \geq 0$. If mechanism is IC, then individually rational implies $u_i(0) \geq 0$, or:

$$q_i(x_i)x_i - m_i(x_i) \geq 0 \iff m_i(0) \leq 0$$

Optimal Mechanism

- The expected payoff of the seller is:

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- The expected payoff of the seller is:

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- The ex-ante expected payment of buyer i is:

$$\begin{aligned} E[m_i(X_i)] = & \int_0^{\omega_i} \left(m_i(0) + q_i(x_i)x_i - \int_0^{x_i} q_i(s_i)ds_i \right) f_i(x_i)dx_i \\ = & m_i(0) + \int_0^{\omega_i} x_i q_i(x_i) f_i(x_i)dx_i \\ & - \int_0^{\omega_i} \int_0^{x_i} q_i(s_i) f_i(x_i) ds_i dx_i \end{aligned}$$

cont.

$$
\begin{aligned}
E[m_i(X_i)] =& m_i(0) + \int_0^{\omega_i} x_i q_i(x_i) f_i(x_i) dx_i \\
& - \int_0^{\omega_i} \int_{s_i}^{\omega_i} q_i(s_i) f_i(x_i) dx_i ds_i \\
=& m_i(0) + \int_0^{\omega_i} (x_i q_i(x_i) f_i(x_i) - q_i(x_i)(1 - F_i(x_i))) dx_i \\
=& m_i(0) + \int_0^{\omega_i} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) q_i(x_i) f_i(x_i) dx_i
\end{aligned}
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cont.

- Using $q_i(z_i) = \int_{\mathcal{X}_{-i}} Q_i(z_i, \mathbf{x}_{-i}) f_{-i}(\mathbf{x}_{-i}) \mathbf{dx}_{-i}$:

$$
\begin{aligned}
E[m_i] &= m_i(0) + \int_0^{\omega_i} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) q_i(x_i) f_i(x_i) dx_i \\
&= m_i(0) + \int_{\mathcal{X}} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) Q_i(x_i) f(\mathbf{x}) \mathbf{dx} \\
E[R] &= \sum_{i \in \mathcal{N}} m_i(0) + \sum_{i \in \mathcal{N}} \int_{\mathcal{X}} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) Q_i(x_i) f(\mathbf{x}) \mathbf{dx}
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- Define $\psi_i(x_i) \equiv x_i - \frac{1-F_i(x_i)}{f_i(x_i)}$. The seller should choose $(\mathbf{Q}, \mathbf{M})$ to maximize:

$$\sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{X}} \left(\sum_{i \in \mathcal{N}} \psi_i(x_i) Q_i(x_i) \right) f(\mathbf{x}) \mathbf{dx}$$

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- We'll focus on these term:

$$\sum_{i \in \mathcal{N}} \psi_i(x_i) Q_i(x_i)$$

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- What do you think?

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- $M_i(0, \mathbf{x}_{-i}) = 0$, and hence, $m_i(0) = 0$.
- The maximized value of the expected revenue is:

$$E[R] = E[\max\{\psi_1(X_1), \psi_2(X_2), \ldots, \psi_N(X_N), 0\}]$$

cont.

- More "intuitive formula":

$$y_i(\mathbf{x}_{-i}) = \inf\left\{z_i : \psi_i(z_i) \geq 0 \ \wedge \ \forall\, j \neq i, \psi_i(z_i) \geq \psi_j(x_j)\right\}$$

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$$Q_i(z_i, \mathbf{x}_{-i}) = \begin{cases} 1, & z_i > y_i(\mathbf{x}_{-i}) \\ 0, & z_i < y_i(\mathbf{x}_{-i}) \end{cases}$$

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$$M_i(\mathbf{x}) = \begin{cases} y_i(\mathbf{x}_{-i}), & Q_i(\mathbf{x}) = 1 \\ 0, & Q_i(\mathbf{x}) = 0 \end{cases}$$

Symmetric Case

- If, for all i, $f_i = f \implies \psi_i = \psi$:

$$E[R] = \max\left\{\psi^{-1}(0), \max_{j \neq i} x_j\right\}$$

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- The optimal mechanism is a second-price auction with a reserve price $r^* = \psi^{-1}(0)$.

References

[1] Vijay Krishna. *Auction Theory*. Elsever Science, 2002.